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AIRPORT RUNWAY CAPACITY AND DELAYS—I

by

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ABSTRACT

When both landing and departing aircraft use a single runway the flow capacity and the delays to aircraft depend on the operating rules which govern use of the runway. In this paper the author analyses those situations where landing aircraft have priority over departures but where departures must satisfy several minimum spacing requirements before they can be interposed between landings. By assuming that inter-arrival times of landing aircraft are independently sampled random variables, from a translated exponential distribution, solutions are obtained for the total movement capacity and the delays to departures at a single runway.

AIRPORT RUNWAY CAPACITY AND DELAYS --- I

1. Introduction

In recent years demands for air service have increased at a rate which often exceeds the rate at which new facilities and new control procedures can be designed and installed to handle these traffic flows. In order to provide economical designs and efficient control procedures, one needs to examine the delay and congestion problems which arise. This is especially true when terminal areas operate near saturation, i.e., as peak flow rates of landing, taxiing and departing aircraft approach the capacity of runway facilities.

The theory of queues and those of stochastic processes in general have suggested many fruitful lines of approach to these delay questions. Since the probability distributions and average delays are extremely sensitive to changes in arrival rates when the servicing facility is close to its capacity, the need for accurate calculations of capacity is evident. Moreover, this theoretical capacity reflects an upper bound on the actual number of landing and departing aircraft which can be serviced per unit time over long periods of time. Once movement capacities have been obtained it is sometimes possible to find delays to landing and departing aircraft. This paper discusses both capacity and delay questions.

Assume that landing aircraft arrive at random over a runway threshold. Under these conditions and in the absence of any departing aircraft, the runway capacity is determined by the length of time an airplane is on the runway. If the service time of the runway (i.e., from "over threshold" to "off runway") is a random variable with mean value, Δ, the steady-state capacity service rate of the runway is Δ^{-1}. These statements are also true when only departing

aircraft are allowed to use the runway. In each case, the capacity or steady-state flow rate through the runway is the reciprocal of the mean service time.

Even if the dominant service times occur on the runway these simple arguments do not extend to mixed operations for several reasons: the flow stream of departing aircraft may not operate independently of the stream of landing aircraft. Interdeparture times may no longer be independently sampled random variables and the runway occupancy times of landing aircraft may differ widely from those of departing aircraft.

In this paper we assume that demand for service is random in the sense that the interarrival times of landing aircraft are independently sampled from known probability distributions. This is not so restrictive as the statement that demand for service is Poisson; on the other hand, it does not allow one to incorporate interarrival times which depend on the history of earlier arrivals.

For the purposes of our analysis, the time scale $t > 0$ can be subdivided into two types of intervals: blocked intervals when a departing aircraft is forbidden use of the runway, and unblocked intervals when he may use the runway. (See Figure 1.)

In Section 2 we obtain the distribution of lengths of blocked and unblocked periods when every point in an unblocked period is removed by more than a constant separation X from a preceding airplane and more than a constant separation Y from a following airplane. In Sections 3 and 4 expressions for the capacity servicing rate of a runway with both landings and take-offs are developed from an asymptotic expansion of the average count of departures interposed in an interval t. Delays to departing aircraft are studied in the final sections.

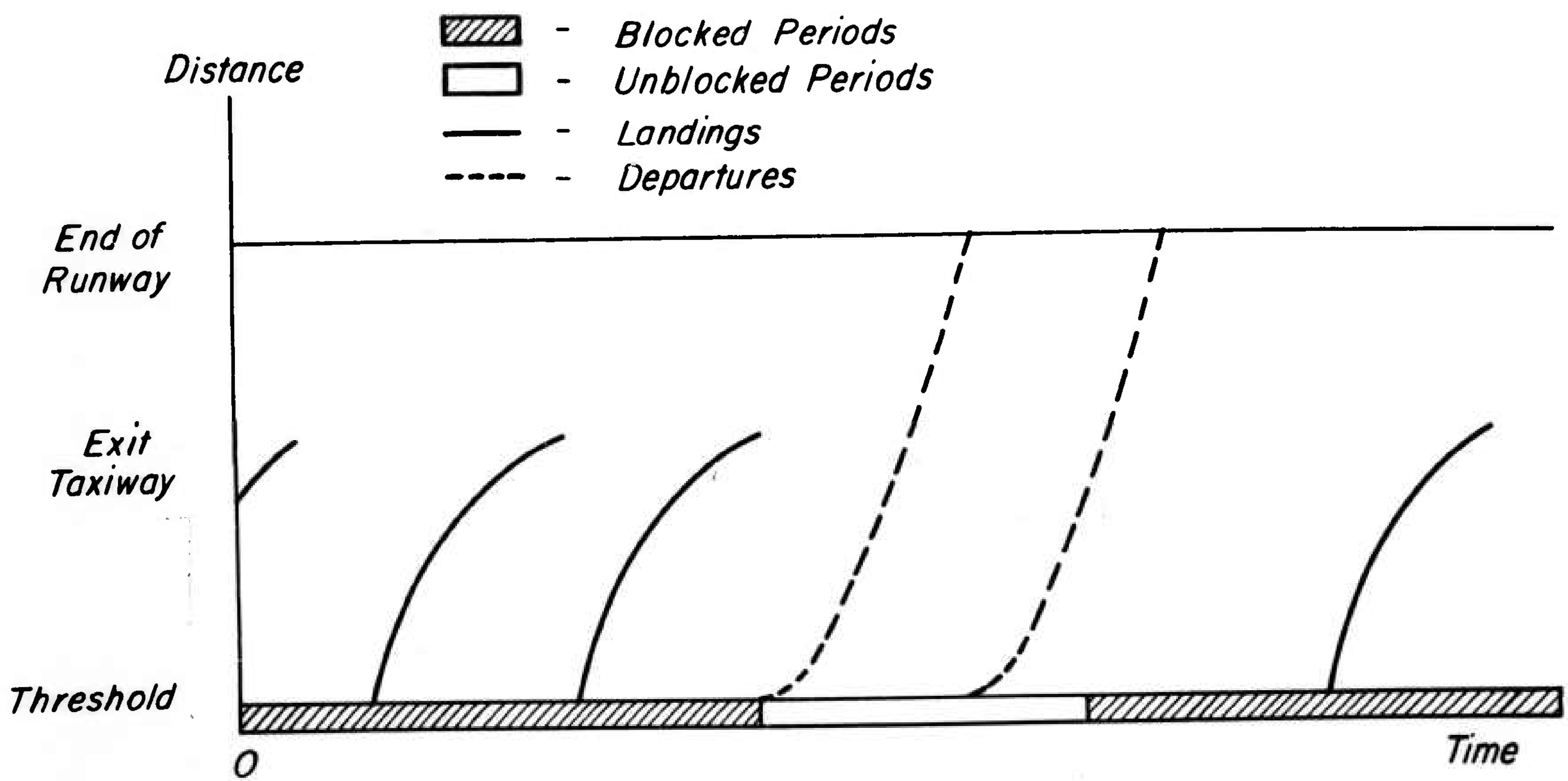

Fig. 1—LANDINGS AND DEPARTURES.

The work of earlier authors is related to the material presented in this paper. Blumstein[1] has discussed the operational capacity of a runway used by landing and departing aircraft. In his models the long-run or steady-state interposition rate of departing aircraft is found when successive departures satisfy a Markov property. The renewal arguments of this paper allow more general interposition rules, explicit expressions for the count of interpositions in a finite interval of time as well as formulas for the delay distributions of departing aircraft.

The result in Section 2 is a slight modification of the ordinary gap and block process in a traffic stream. The reader is referred to the papers by Tanner[9], Weiss and Maradulin[10], and Oliver[5], and references contained therein for more detailed treatment of this well-known problem.

The delays in the queue of departing aircraft is similar to a paper by Tanner[8] dealing with the interactions of two opposing Poisson streams of traffic. Interference and queues arise when there is simultaneous demand for service at a facility that can serve only one stream at a time. In Tanner's paper, priority is determined by the first occupant of the facility. Recent work by Gaver[2] obtains some important results for priority queueing models where customers queue for gaps in a primary traffic stream. Analogies between his and our model are discussed in the text.

The notation used in this paper is similar to that used in an earlier paper on the distribution of blocks and unblocked periods in a traffic stream.[5] Interevent density distributions are denoted by small letters. Capital letters denote the probability that the random variable is greater than or equal to the argument. Average values and variances of the density distributions are shown by ν and σ^2 with subscripts referring to the appropriate density distribution.

λ and μ, with or without subscripts, refer to steady-state arrival and service rates. Laplace transforms are denoted by a tilde (~) over the function being transformed and $\delta(t)$ is the dirac delta function. Subscripts d and ℓ refer to departing and landing aircraft respectively.

2. Blocked and Unblocked Periods

In order to study the distribution of blocked periods in which departing aircraft are forbidden use of the runway, we assume that aircraft pass over the threshold of a runway at instants of time $\tau_0, \tau_1, \tau_2, \tau_n, \ldots$ and that the intervals $t_n = \tau_{n+1} - \tau_n$ between arrivals are independently sampled from a common density distribution $a(t)$ with mean value ν and variance σ^2.

If $\tau_0 = 0$ and n aircraft pass over the threshold of the runway, the last one arriving at the instant τ_n, an "arbitrary" point has the probability $dt \cdot \tau_n^{-1}$ of landing in an interval dt. The arbitrary point must necessarily land in an interval which is bounded on the left-hand side by some τ_j and on the right by some τ_{j+1}; $0 \leq j \leq n-1$. Since the arbitrary point is more likely to be covered by long than by short interarrival intervals, the probability density distribution, $v(t)$, of the interarrival time which covers the arbitrary point is proportional to t and $a(t)$ as n becomes large[6]:

$$v(t) = \nu^{-1} t a(t) \tag{1}$$

This distribution has mean and variance

$$\nu_v = \frac{\nu^{(2)}}{\nu} ; \tag{2}$$

$$\sigma_v^2 = \nu^{-1}\nu^{(3)} - \nu^{-2}(\nu^{(2)})^2$$

where $\nu^{(n)}$ is the n^{th} moment (about zero) of $a(t)$.

The conditional probability density distribution $u(r/t)$ that the spacing from an arbitrary point to the following (or preceding) aircraft is between r and $r + dr$ given that the point is covered by an interval of length t is the

rectangular distribution over $(0, t]$

$$u(r/t)dr = \frac{dr}{t} \qquad\qquad 0 < r \leq t \tag{3}$$

and the marginal density distribution of spacings from the arbitrary point to either a following or preceding airplane is the "starting-at-random" or "next arrival" density* $u(t) = \nu^{-1}A(t)$ which has the following physical interpretation: If one arrives at the runway threshold at an instant of time which is uncorrelated with the arrival times $\{\tau_n\}$ the probability density distribution of wait to the first airplane which passes over the runway threshold is $u(t)$. The conditional probability density distribution $v(t/r)$ that the arbitrary point is covered by an interval of length t given the spacing r to the first following (or preceding vehicle) is therefore

$$v(t/r) = \frac{u(r/t)v(t)}{\nu^{-1}A(r)} = \frac{a(t)}{A(r)}, \qquad\qquad r \leq t \tag{4}$$

Define unblocked points as the collection of points which are separated by more than an interval Y from the first preceding airplane and more than X from the first following airplane. All other points on the real axis are blocked. The probability that an arbitrary point will land in an unblocked period is the joint probability that the spacing to the vehicle on the left is greater than or equal to X and that the spacing to the vehicle on the right is greater than or equal to Y. Hence, from the definition and Equation (4) we get

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* Reference 7, p. 12.

$$\Pr\left\{\text{arbitrary point lands in an unblocked period}\right\}$$

$$= \int_X^\infty \int_Y^\infty v(r + t/t)\, u(t) dr dt \tag{5}$$

$$= U(X + Y)$$

In a traffic context this probability is called the transparency of the landing stream, i. e., the fraction of time the landing stream can accept departures. The probability, $V(t)$, that an arbitrary point is covered by an interarrival time greater than or equal to t, is greater than $U(t)$; in the former case, we allow one vehicle to be arbitrarily close to the arbitrary point while we do not in the latter case. In general we find that

$$V(t) - U(t) = tu(t) > 0$$

and it is only in the Poisson case (exponentially distributed interarrival times) that the probability of landing in an unblocked period is equal to the product of the independent probabilities that the spacing to the preceding airplane is greater than Y and to the following is greater than X.

When departing aircraft must meet minimum spacing requirements to preceding and following landings as well as departures, the length of a blocked period consists of the sum of three parts: (i) an interval X which precedes the arrival of the first of several landing aircraft, each with interarrival time less than $X + Y$; (ii) the length of an interval from this first arrival to the first arrival which begins a gap greater than $X + Y$; and (iii) an interval Y which follows the last mentioned arrival. Each unblocked period must be covered by an interarrival headway between landing aircraft which is greater than or equal to $Z = X + Y$. Clearly, the distribution of lengths of unblocked periods does not

depend on the location of the unblocked period between two arrivals but only on the distribution of intervals which exceed Z. The density distribution of lengths of unblocked periods, $h(t)$, has a transform[5]

$$\tilde{h}(s) = e^{sZ}A^{-1}(Z)[\tilde{a}(s) - \tilde{a}(s;Z)] \tag{6}$$

where $\tilde{a}(s)$ is the transform of $a(t)$ and $\tilde{a}(s;Z) = \int_0^Z a(t)e^{-st}dt$. We denote by $f(t)$ the density distribution of lengths of blocked periods, i.e., those points which lie outside the minimum spacing requirements to adjacent aircraft. It was also shown that this density distribution has a Laplace transform

$$\tilde{f}(s) = \frac{e^{-sZ}A(Z)}{1 - \tilde{a}(s;Z)} \tag{7}$$

For future reference we also note that the first and second moment of $f(t)$ can be written in the form

$$\nu_f = \nu A^{-1}(Z)(1 - U(Z)) \tag{8}$$

$$\nu_f^{(2)} = 2\nu_f^2 - 2\nu_f Z + Z^2 + A^{-1}(Z)\left\{\nu^{(2)} - \int_Z^\infty t^2 a(t)dt\right\} \tag{9}$$

3. Interpositions

It should be obvious from a practical as well as a theoretical viewpoint that the number of departing aircraft which can merge into the unblocked periods of the stream of landing aircraft, is not equal to the difference between a capacity service rate of the runway and the observed flow rate of landing aircraft. While this computation may serve as an approximation in certain situations, the simple analysis breaks down for reasons we have already mentioned. Departing aircraft may only be able to merge with the stream of landing aircraft when certain minimum separations are maintained between adjacent landing as well as departing aircraft.

To obtain the counting distribution of departing aircraft which can be interposed between landing aircraft, we consider the hypothetical situation where an infinite queue of departing aircraft waits at the side of the runway. When an unblocked period appears, a departure is interposed, i.e., is allowed to take off. If the unblocked period is long enough, there may be several interpositions. In no case, however, are departing aircraft allowed to come arbitrarily close to one another.

Interpositions can be counted in several ways. For example, one can specify intervals which begin, (i) just after a landing has crossed the runway threshold, (ii) just after a departure is released, (iii) at random, (iv) at the beginning of an unblocked period, or (v) at the beginning of a blocked period. The count depends not only on the location of this interval, but also on the rules which govern the precise instants when departing aircraft are released into the primary stream.

In the remainder of this paper we focus attention on an interval (0, t) which begins with either blocked or unblocked periods. If we start to count

at the instant when a blocked period begins, the first departure is released at the instant that the blocked period ends, i.e., the unblocked period begins. No departures will be released in an interval (0, t) beginning with a blocked period if the blocked period is greater than t. If we let

$$p_n(t) = \Pr \begin{Bmatrix} \text{n interpositions in } [0, t) \text{ starting at} \\ \text{the beginning of a blocked period} \end{Bmatrix} \tag{10}$$

we have the special case,

$$p_0(t) = \int_t^\infty f(x)\,dx = F(t) \tag{11}$$

If the first blocked period ends at an instant r such that $r < t$, the count in $(0, t)$ equals the count in $[r, t)$. But the count in $[r, t)$ is the count one would observe in an interval $[0, t-r)$ whose origin was located at the beginning of an unblocked period. Defining $q_n(t)$ as the discrete counting distribution of departures when the counting origin starts at the beginning of an unblocked period, we have

$$p_n(t) = \int_0^t f(r) q_n(t-r)\,dr \quad . \qquad n \geq 1 \tag{12}$$

In order to obtain a second renewal equation expressing $q_n(t)$ in terms of $p_n(t)$ we consider the events which lead to a count of n if the origin is located at the beginning of an unblocked period. To do this we use still another conditional probability distribution. If an unblocked period is of length r, we define

$$h(m|r) = \Pr \begin{Bmatrix} \text{m interpositions} \,|\, \text{unblocked} \\ \text{interval of length } r \end{Bmatrix} \tag{13}$$

and assume that $h(m|r)$ is 1 if operating rules allow m interpositions in r or 0 if they do not. A total count of n interpositions in the interval $[0, t)$ starting with an unblocked period of length r can be observed if $m < n$ are observed in $[0, r)$ and $(n-m)$ are observed in $[r, t)$. A count of n will also be observed in t if $r > t$ and n departing aircraft can be interposed in $[0, t)$. The probability of n interpositions, is the sum, over all m, of the joint probabilities discussed above. Hence,

$$q_n(t) = \int_0^t \sum_{m=1}^{n} h(m|r)h(r)p_{n-m}(t-r)dr + h(n|t)H(t) \quad . \qquad n \geq 1 \tag{14}$$

The solutions of $q_n(t)$ (or $p_n(t)$) can be written as a generating function in terms of

$$\tilde{h}(z;s) = \sum_{n=1}^{\infty} \int_0^{\infty} h(n|t)h(t)e^{-st}z^n dt \quad . \tag{15a}$$

$$\tilde{H}(z;s) = \sum_{n=1}^{\infty} \int_0^{\infty} h(n|t)H(t)e^{-st}z^n dt \quad . \tag{15b}$$

Multiplying both sides of Equation (14) by z^n and summing over n gives

$$\tilde{Q}(z;s) = \sum_{n=1}^{\infty} \int_0^{\infty} q_n(t)e^{-st}z^n dt = \left[\tilde{h}(z;s)(1-\tilde{f}(s)) + s\tilde{H}(z;s)\right]\left[s - s\tilde{f}(s)\tilde{h}(z;s)\right]^{-1} \tag{16}$$

The analagous transform for $\tilde{P}(z;s)$ is obtained by multiplying $\tilde{Q}(z;s)$ by $\tilde{f}(s)$ and adding $\tilde{p}_0(s)$.

4. Capacity Service Rates

In the remainder of this paper we consider capacity interposition rates and queueing effects under the specific assumption that a headway between two landings exceeds t has probability

$$A(t) = 1 \qquad 0 \leq t < \Delta_\ell$$

$$= e^{-\lambda_\ell (t-\Delta_\ell)/1-\lambda_\ell \Delta_\ell} \qquad \Delta_\ell \leq t \tag{17}$$

When λ_ℓ is the average arrival rate of landing aircraft and Δ_ℓ is the minimum headway between landings, this distribution has been found to be a reasonably accurate description of aircraft movements over threshold.[4] We observe, in passing, that $\Delta_\ell = 0$ corresponds to the case of Poisson arrivals.

In order to construct the blocked and unblocked periods in the stream of landing aircraft we now specify three constants $\Delta_{d\ell}$, $\Delta_{\ell d}$ and Δ_d: A departing airplane on the runway must at all times maintain a headway greater than or equal to $\Delta_{d\ell}$ ahead of a following landing and a headway greater than or equal to $\Delta_{\ell d}$ behind an earlier landing. In addition, two departures located within a common unblocked period must maintain at least the headway Δ_d between each other. If we now assume (Figure 1) that the minimum size blocked period must equal the sum of these three constants, algebraic simplifications in our formulas are obtained by setting

$$\Delta = \Delta_{d\ell} + \Delta_{\ell d} + \Delta_d$$

It may happen that $\Delta_\ell > \Delta$ in which case there can be one or more inter-

positions between every landing; we assume, however, that $\Delta > \Delta_\ell$. Since $\Delta > \Delta_d$ it is also obvious that two departures located in adjacent unblocked periods do not interfere with one another's movements.

Substituting the transforms

$$\tilde{a}(s) = \lambda_\ell e^{-s\Delta_\ell}\left(\lambda_\ell + s(1-\lambda_\ell\Delta_\ell)\right)^{-1}$$

$$\tilde{a}(s;\Delta) = \lambda_\ell\left(\lambda_\ell + s(1-\lambda_\ell\Delta_\ell)\right)\left(e^{-s\Delta_\ell} - e^{-s\Delta}\, e^{-\frac{\lambda_\ell(\Delta-\Delta_\ell)}{1-\lambda_\ell\Delta_\ell}}\right) \tag{18}$$

into Equation (6) leads to the exponential distribution,

$$h(t) = \frac{\lambda_\ell}{1-\lambda_\ell\Delta_\ell}\ \exp - (\lambda_\ell t/\ 1 - \lambda_\ell\Delta_\ell) \tag{19}$$

for the length of unblocked periods. If the first departure is interposed at the beginning of an unblocked period and an interposition is made every Δ_d time units thereafter within the same unblocked period, the transform $\tilde{h}(z;s)$ in Equation (15a) can be written

$$\lambda H(z;s) = \tilde{h}(z;s) = \frac{z\lambda}{\lambda+s}\left[1 - e^{-(\lambda+s)\Delta_d}\right]\left[1 - ze^{-(\lambda+s)\Delta_d}\right]^{-1} \tag{20}$$

where $\lambda = \lambda_\ell(1 - \lambda_\ell\Delta_\ell)^{-1} > \lambda_\ell$.

The Laplace transform of the average count of interpositions,

$$\tilde{M}(s) = \int_0^\infty M(t)e^{-st}dt = \int_0^\infty \sum_{n=0}^{\infty} nq_n(t)e^{-st}dt\ , \tag{21}$$

equals the partial derivative of $\tilde{Q}(z;s)$ in Equation (16), with respect to z at $z = 1$.* Substituting Equation (6), (18) and (20) into Equation (16) leads, asymptotically, to the expression for the capacity interposition rate of departing aircraft:

$$\mu_d = \lim_{t\to\infty} \frac{1}{t} M(t) = \lim_{s\to 0} s^2\tilde{M}(s) \tag{22}$$

$$= \lambda_\ell\left[\exp - (\lambda_\ell(\Delta - \Delta_\ell) / 1 - \lambda_\ell\Delta_\ell)\right]\left[1 - \exp - (\lambda_\ell\Delta_d / 1 - \lambda_\ell\Delta_\ell)\right]^{-1}$$

Since the partial derivative of $\tilde{h}(z;s)$ with respect to z at $z = 1$, $s = 0$ equals the average number of interpositions per unblocked period and the average flow rate of unblocked periods equals $\mu A(\Delta)$, μ_d can be written as the product of such terms, an observation which seems intuitive in retrospect but difficult to prove in individual cases. However, these results hold asymptotically for a broad class of statistical interposition rules described by Jewell.[3]

For small λ_ℓ, the capacity interposition rate,

$$\mu_d = \Delta_d^{-1}\,[1-\lambda_\ell(\Delta-\Delta_d/2)] + 0(\lambda_\ell^2)\,, \tag{23}$$

is equal to the capacity service rate for a "departures-only" runway operation minus a small term which is linear in λ_ℓ, the average arrival rate of landings.

Figure 2 is a plot of μ_d for several values of Δ_ℓ, Δ_d and Δ which are encountered in typical runway operations with jet and propellor powered aircraft in the U.S.A.

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* A similar transform can be obtained for the counting distribution $p_n(t)$.

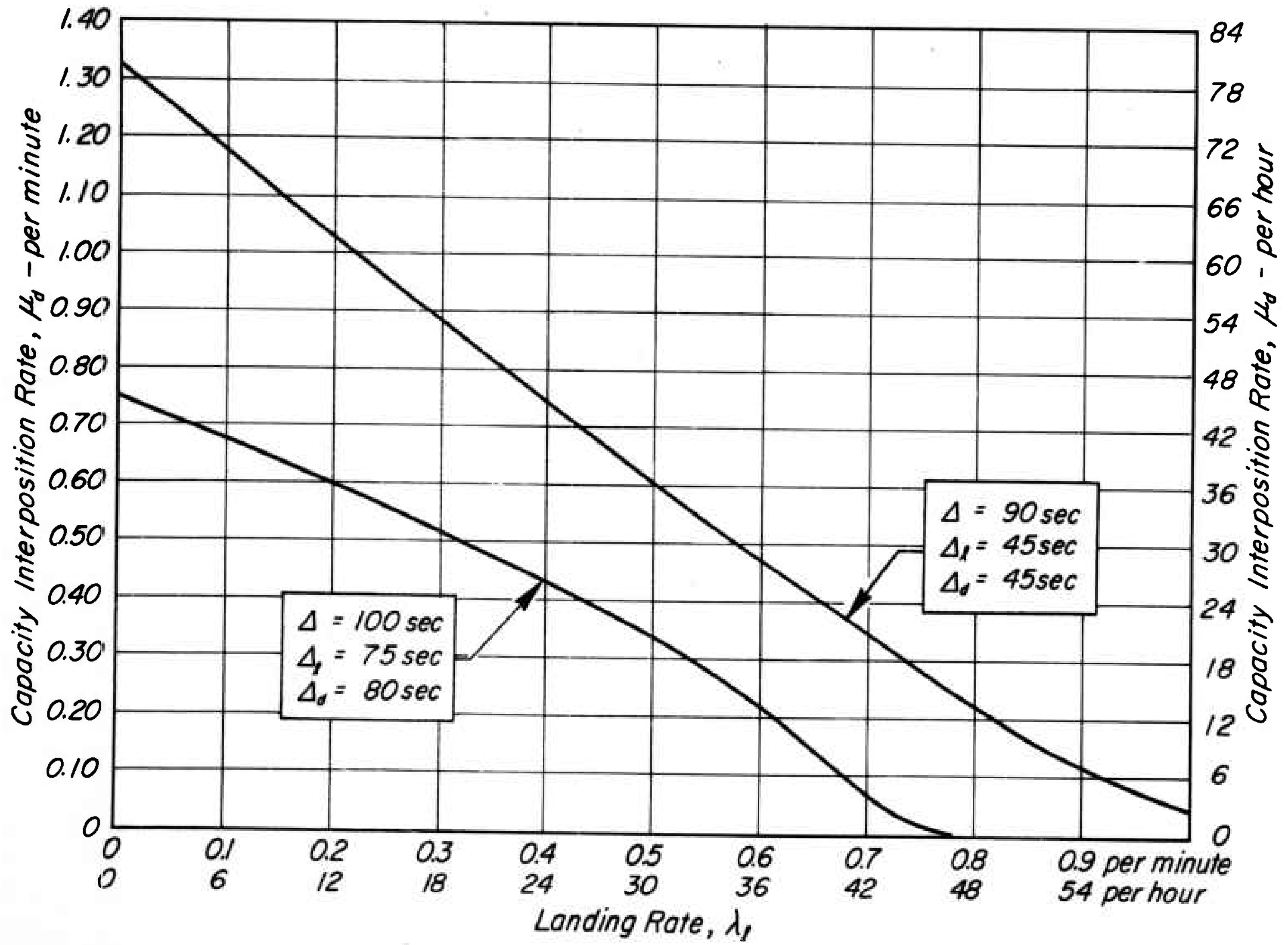

Fig.2 – CAPACITY INTERPOSITION RATE.

5. Delays to Departing Aircraft

Once the arrival pattern of landing aircraft and the capacity interposition rate of departing aircraft is known, it is possible to obtain an expression for the distribution of delays to departing aircraft. In particular, we are interested in the distribution of elapsed time from arrival to interposition. With some differences, which we shall discuss shortly, these types of delay problems can be studied under the heading of priority queueing problems where the lower priority (departing aircraft) must wait for unblocked periods to appear in the high priority stream before service can begin.

If we focus attention on those instants of time when a departure begins use of the runway, the headway to the earliest succeeding departure (in the absence of landings) is Δ_d. Since the length of unblocked periods (Equation (19)) is exponential with constant $\lambda = \lambda_\ell / 1-\lambda_\ell\Delta_\ell > \lambda_\ell$, the probability that the headway from the beginning of service to the beginning of the blocked period exceeds Δ_d is $e^{-\lambda\Delta_d}$. Hence this is the probability that another departure precedes a landing, i.e., the service time of the second departure is also Δ_d.

On the other hand, the probability that at least one landing precedes another departure is $1-e^{-\lambda\Delta_d}$. In this case the headway to the first following departure is

$$S = (S_0) + (S_1 + S_2 + \ldots) + (\Delta) \qquad (24)$$

where S_0 is the headway $(<\Delta_d)$ to the beginning of the unblocked period, $S_i (i \geq 1)$ are those headways between landing aircraft which lie within a blocked period and the last term is the sum of the minimum clearances between adjacent aircraft.

The probability density of S_0 is

$$\Pr\left\{S_0 = t\right\} = \frac{\lambda e^{-\lambda t}}{1-e^{-\lambda\Delta_d}} \qquad 0 \leq t < \Delta_d \tag{25}$$

and since the arrangement of remaining terms in Equation (24) is not important, the probability distribution of $S - S_0$ is the distribution of lengths of blocked periods. The density distribution $d(t)$ of the headway between successive departures is now obtained by removing the condition on the length of S_0 and adding the probability that the service time is exactly Δ_d:

$$d(t) = e^{-\lambda\Delta_d}\,\delta(t-\Delta_d) + \int_0^{\Delta_d} \lambda e^{-\lambda r} f(t-r)dt \tag{26}$$

We again point out that the blocked periods are constructed from a landing stream with interarrival headways given by Equation (17). Substituting Equation (18) into Equation (7) for $\tilde{f}(s)$ and collecting terms, we obtain the Laplace transform of the probability that the service time of a departure exceeds t as,

$$\tilde{D}(s) = \int_0^\infty D(t)e^{-st}dt = \frac{\left(\lambda + s - \lambda e^{-s\Delta_\ell}\right)\left(1 - e^{-(\lambda+s)\Delta_d}\right)}{s\left(\lambda + s - \lambda e^{-s\Delta_\ell} + \lambda e^{-(\lambda+s)\Delta}\, e^{\lambda\Delta_\ell}\right)} \tag{27}$$

As $s \to 0$ we obtain the mean service time of departures

$$\nu_d = \lim_{s\to 0} \tilde{D}(s) = \lambda_\ell^{-1} e^{\lambda(\Delta-\Delta_\ell)}\left(1 - e^{-\lambda\Delta_d}\right) \tag{28}$$

which is just the reciprocal of the capacity service rate in Equation (22).

The expected delays to departing aircraft can be obtained by direct renewal arguments when we assume that arrivals to the departure queue are Poisson with mean flow rate λ_d and that first-come, first-serve rules are observed for departures. Let π_0 be the probability that there are no departures on the runway or waiting to be interposed between departures. Since, for long run equilibrium conditions, the expected number of departures demanding service must equal the expected number interposed in the landing stream, we have

$$\lambda_d t = \mu_d(1-\pi_0)t + 0(1)$$

Hence as $\quad t \to \infty$ (29)

$$\pi_0 = 1 - \lambda_d \nu_d .$$

A departure arriving at an arbitrary instant of time waits for use of the runway; this random wait has a density distribution $w(t)$. If the arrival finds one or more departures waiting, the probability density of his wait equaling t is

$$\mu_d \int_0^t D(r)w(t-r)dr \qquad t > 0 \tag{30}$$

since $\mu_d D(r)$ is the probability density that the departure at the head of the queue must wait a time r for use of the runway. If a departure arrives when no other departures are waiting, he may occupy the runway at once, i.e., the wait is zero, or he may have to wait until the current blocked period, of minimum size Δ, passes. In the former case, the probability of landing in an unblocked period is, by a long run argument, equal to the

fraction of time occupied by unblocked periods, i.e., the transparency of the stream,

$$U(\Delta) = (1 + \lambda\nu_f)^{-1} \tag{31}$$

while the complementary probability is the chance of landing in a blocked period. If the departure arrives during a blocked period the probability density of headways to the end of the blocked period is $\nu_f^{-1}F(t)$. Adding up the probabilities of exclusive events which lead to a wait of t, we finally get the density distribution of wait for use of the runway:

$$w(t) = \frac{1-\lambda_d\nu_d}{1+\lambda\nu_f}\left[\delta(t) + \lambda F(t)\right] + \lambda_d \int_0^t D(r)w(t-r)dr \,. \tag{32}$$

The Laplace transform of Equation (32),

$$\tilde{w}(s) = \frac{1+\lambda\tilde{F}(s)}{1-\lambda_d\tilde{D}(s)} \quad \frac{1-\lambda_d\nu_d}{1+\lambda\nu_f} \tag{33}$$

is a modification of the Pollaczek-Khintchine formula[7] which accounts for the fraction of time when departures arrive too close to landings for safe take-off operations.

Gaver[2], has obtained a similar result for the waiting time distribution of vehicles which queue for gaps between blocked periods. His blocked periods would have (in our runway model) a minimum size $\Delta - \Delta_d = \Delta_{d\ell} + \Delta_{\ell d}$; lying between each blocked period is a gap which can be used by one (or more) departing aircraft provided its size exceeds Δ_d, the minimum headway required by departures. As a pre-emptive priority repeat

queueing model, he finds that a departure's "completion-time" consists of an alternating sequence of pre-emptions by landings (blocked periods) and insufficient gaps (unblocked periods $< \Delta_d$) before it is actually allowed onto the runway.

Let $\tilde{b}(s)$ be the transform of the blocked period distribution obtained when interlanding times are given by Equation (17) and we substitute $\Delta_{d\ell} + \Delta_{\ell d}$ for Z in Equation (6) and (7). By using the identity

$$\tilde{f}(s) = \frac{e^{-(s+\lambda)\Delta_d}\,\tilde{b}(s)}{1 - \frac{\lambda\tilde{b}(s)}{\lambda + s}\left[1 - e^{-(\lambda+s)\Delta_d}\right]} \tag{34}$$

it is possible to obtain Gaver's result,*

$$\tilde{w}(s)e^{-s\Delta_d}\left[\tilde{d}(s)\right]^{-1} = \frac{1 - \lambda_d\nu_d}{1 - \lambda_d\tilde{D}(s)} \cdot \frac{1 + \lambda\tilde{B}(s)}{1 + \lambda\nu_b}, \tag{35}$$

for the transform of the waiting time that elapses from the instant a departure joins the queue until it is at the head of the line.

The expected delay, w_d, in the departure queue can be obtained by direct integration of Equation (32):

$$w_d = \frac{\lambda_d\nu_d^{(2)}}{2(1-\lambda_d\nu_d)} + \frac{\lambda\nu_f^{(2)}}{2(1+\lambda\nu_f)} \tag{36}$$

As we have already mentioned, the general characteristics of the Pollaczek-Khintchine formula are apparent in the first term. The second term is the

* Reference [2], Section 7.

additional contribution of blocked periods to average delay of departures. One might be tempted to conclude that the effect of the blocked periods is small, since the major contribution to average delay (with high departure flow rates) is due to the left hand term in (36) involving only the first and second moment of the service time distribution. However, it must be remembered that both ν_d and $\nu_d^{(2)}$ depend upon the length of blocked periods. (Equations (26) and (27)).

Numerical calculations can now be completed by substituting the moments of the service time distribution in Equation (26) and moments of blocked period lengths which can be obtained by substituting Equation (17) into Equations (8) and (9). Since the algebraic expressions are involved, we now turn to low flow rate approximations and the special case of Poisson landings and departures.

6. Low Flow Approximations

As either the arrival rate of landing or departing aircraft becomes small, the expressions for capacity servicing rates and delays simplify considerably. For low arrival rates of landing aircraft, $\nu_d \to \Delta_d$ and $\nu_d^{(2)} \to \Delta_d^2$. The transform of the waiting distribution (Equation (33)) approaches that of a constant service queue since the restriction on immediate interposition of an arbitrary departure is due to the minimum separation Δ_d between departures but not to the higher priority landing aircraft. In particular, the average delay to departing aircraft becomes

$$\lim_{\lambda_\ell \to 0} w_d = \frac{\lambda_d \Delta_d^2}{2(1-\lambda_d \Delta_d)} . \tag{37}$$

On the other hand, as the average arrival rate of departures fall off to zero, departure delays are primarily determined by the presence or absence of landing aircraft while the effect of interference between departures is small. Expansion of Equation (33) gives

$$\begin{aligned} \tilde{w}(s) &= \left[\frac{1+\lambda \tilde{F}(s)}{1+\lambda \nu_f}\right] (1+0(\lambda_d)) \\ &= \left[\frac{e^{-\lambda(\Delta-\Delta_\ell)}(\lambda+s)\left(\lambda+s-\lambda e^{-s\Delta_\ell}\right)}{s(1+\lambda\Delta_\ell)\left(\lambda+s-\lambda e^{-s\Delta_\ell}+\lambda e^{-s\Delta} e^{-\lambda(\Delta-\Delta_\ell)}\right)}\right] (1+0(\lambda_d) . \end{aligned} \tag{38}$$

The term in square brackets is the transform of the wait for an unblocked period. The limiting value of the average delay to departing aircraft is therefore equal to the average wait for an unblocked period in the landing stream.*

---

* Equation (42), p. 207, and Equation (47b), p. 208, Reference [5].

When landings are Poisson, $\Delta_\ell = 0$ and $\lambda = \lambda_\ell$ in Equation (17). If we also assume that $\Delta_{d\ell}$ and $\Delta_{\ell d}$ are small compared to Δ_d, the blocked periods and the time between interposition of departures have identical distributions with average value and second moment equal to

$$
\begin{aligned}
\nu_d &= {}_d = \lambda_\ell^{-1}\left(e^{\lambda_\ell \Delta_d} - 1\right) \\
\nu_f^{(2)} &= \nu_d^{(2)} = 2\lambda_\ell^{-2}\, e^{\lambda_\ell \Delta_d}\left[e^{\lambda_\ell \Delta_d} - \lambda_\ell \Delta_d - 1\right]
\end{aligned}
\tag{39}
$$

Substituting these expressions into Equation (36) gives

$$
w_d = \frac{\lambda_\ell + \lambda_d}{\lambda_\ell \lambda_d} \left[\frac{e^{\lambda_\ell \Delta_d} - \lambda_\ell \Delta_d - 1}{1 + \lambda_\ell \lambda_d^{-1} - e^{\lambda_\ell \Delta_d}}\right] \tag{40}
$$

for the average delay to departures. For positive values of λ_d less than the capacity service rate, the term in square brackets is also positive and a plot of w_d as a function of λ_d has the characteristic shape of the delay to arrivals in a single channel queue.

7. Summary

The analysis of aircraft service rates and delays has focused on mixed runway operations where low priority departures are interposed between high priority landing aircraft and where minimum separation standards are required between all aircraft. Capacity interposition rates of the servicing system have been found, since they yield an upper bound to the interposition or total movement rate which can be sustained over long periods of time. Even in the absence of priority queueing models which incorporate specific holding patterns for landings or departures, one can expect to find measures of cost and delay built around such capacity figures. In addition, they can also be used to assess the trade-off between landing and departure service rates.

Actual operations on a runway are complicated by several factors which we did not include. In the first place, the concave (convex) time-distance trajectories of landings (departures) may begin or terminate at several rather than one point due to the introduction of multiple taxiways. The shape of these curves will also depend on the type of taxiways, landing speeds and accellerating and braking characteristics of aircraft.

In general, the time (or distance) an airplane occupies the runway will be a random variable. As the real-time control of runway operations becomes more precise through the use of semi-automatic surveillance and computing equipment, minimum separation standards may depend upon aircraft type within a priority class rather than be constant as we have assumed. In addition, simultaneous minimum distance and headway criteria may be imposed.

We have also assumed that delays to congestion are caused by runway operations. As aircraft flight characteristics and instrument flight rules change, dominant service operations may appear in the glide path or along the airways which lead into a terminal area rather than on the ground. However, one believes intuitively that many of the results obtained in these simpler models will extend to those which include the effects mentioned above.

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